2.2.12. Theorem. (Matrix Tree Theorem) Given a loopless graph G with vertex set $v_1, \ldots, v_n$, let $a_{i,j}$ be the number of edges with endpoints v_i and v_j. Let $Q(G)$ be the matrix $D - A$, where A is the adjacency matrix of G and D is the diagonal matrix with $d(v_i)$ in position (i, i) for each i. If $Q^*(G)$ is the matrix obtained by deleting row s and column s of $Q(G)$, then $\tau(G) = \det Q^*(G)$.

Proof: (Bollobás [1998, p. 57]) We use induction on $e(G)$. If $e(G) = 0$ and $n(G) > 1$, then G has no spanning tree and $Q^*(G)$ is an all-zero matrix of order at least 1, which has determinant 0. If $e(G) = 0$ and $n(G) = 1$, then G has one spanning tree and $Q^*(G)$ is a 0-by-0 matrix, which by convention has determinant 1.

Now consider $e(G) > 0$. By renumbering vertices, we may assume that $s = 1$ and that $v_1 \leftrightarrow v_2$. Let e be an edge with endpoints v_1 and v_2, and let $d_i = d_G(v_i)$. The matrices $Q(G - e)$ and $Q(G \cdot e)$ are similar to $Q(G)$. In fact, they all have the same submatrix in the last $n(G) - 2$ rows and columns; call this submatrix R. Let P denote the part of the second row of $Q(G)$ after the second column; this part of $Q(G - e)$ is the same. Let d' denote the degree in $G \cdot e$ of the contracted vertex, and let P' be the remainder of its row in $Q(G \cdot e)$.

By the induction hypothesis, $\tau(G - e) = \det Q^*(G - e)$ and $\tau(G \cdot e) = \det Q^*(G \cdot e)$. By Proposition 2.2.8, these sum to $\tau(G)$. Let $d_i^- = d_i - 1$, and let $a = -a_{1,2}$ and $a^- = -(a_{1,2} - 1)$. The steps in the computation appear below. In each matrix, the row and column deleted before taking the determinant are shaded.

$$\tau(G) = \tau(G - e) + \tau(G \cdot e) = \det Q^*(G - e) + \det Q^*(G \cdot e)$$

$$= \begin{array}{|c|c|c|} \hline d_1^- & a^- & \\ \hline a^- & d_2^- & P \\ \hline & P^T & R \\ \hline \end{array} + \begin{array}{|c|c|} \hline d' & P' \\ \hline P'^T & R \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline d_1 & a & \\ \hline a & d_2 & P \\ \hline & P^T & R \\ \hline \end{array}$$

$$= \det Q^*(G).$$

The equality in the determinant computation uses the expansion formula along the row containing P. The terms involving P contribute the same to $\det Q^*(G - e)$ and $\det Q^*(G)$, since the entry in P is the same and the entries below this row are the same. For the first position in this row, the computation from $G - e$ yields $(d_2 - 1) \det R$. Added to this is the contribution $\det R$ from $G \cdot e$. Hence the sum of all the contributions is precisely equal to $\det Q^*(G)$, as desired. ■

The full statement of the Matrix Tree Theorem allows deletion of any row and column: $\tau(G) = (-1)^{s+t} \det Q^*$ when the submatrix obtained by deleting row s and column t from Q is Q^*. This follows from a lemma in linear algebra stating that when every row and column has sum 0, the cofactors are all equal; see Exercise 8.6.18.